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The Composite Electron

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In this paper, the electron is considered a bound state of a neutrino and a negative pion. A model Lagrangian density that combines weak and electromagnetic interactions gives rise to equations of motion that define such a state. In this model, the muon is a bound state of an antineutrino and a negative pion, which explains why it cannot decay into an electron and a photon. The decay of unstable particles is reduced to pair creation plus particle recombination. The neutral pion is described by an interference between the charged-pion states. Several variations of the model are also presented.

1. INTRODUCTION

In the half century since the beginning of quantum electrodynamics, great progress has been made in the understanding of the interactions between charged particles and the electromagnetic field. Furthermore, the weak interactions between particles have been unified with the electromagnetic ones, and progress has been made in further unification with strong and even gravitational interactions.

Nevertheless, in spite of impressive agreement between theory and experiment, many physicists share the opinion that the ultimate formulation of these theories has to be different from the present one in significant ways. The whole renormalization procedure that underlies the calculations is ill defined from a mathematical point of view. The special role that time plays in particle physics is not always fully appreciated. In relativistic quantum mechanics, equations may be covariant but the time-boundary conditions generally are not. Time is a parameter that marks the development of a state, while the space variables are continuous indices that label the degrees of freedom of the system. A particular Lorentz frame represents an observer at rest in that frame, and that observer determines the type of initial and

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final conditions that are imposed on the system. The special role of time in quantum field theory is quite obvious in the Schrödinger picture, less so in the "manifestly covariant" Heisenberg picture, with the Dirac picture in between. The Gupta–Bleuler formalism, which singles out the time component of the electromagnetic potentials, is an awkward way of handling the constraints in the electromagnetic theory that is best avoided in favor of other approaches. Other long-standing puzzles remain. Where do the masses of the leptons come from, and why are the masses of the electron and the muon so different? Why have no electrically charged massless particles been observed? Why are there two (or more) neutrinos and in what sense do they differ? What is the matter–antimatter distribution in the universe, and why? Are there simpler and more elegant theories of "elementary" particles?

The research that underlies the model presented here is guided by the belief that a better theory will be similar, but not equivalent, to the theory that is now generally accepted. Even if different formulations ultimately turn out to be equivalent, it is often useful to have several alternative views of theoretical and experimental results to help in specific applications.

We propose a new formulation of the electromagnetic and weak interactions. We focus our attention on the older leptons and examine only the most basic facts. Although we choose a specific Lagrangian density to describe the system, other Lagrangians can accommodate the same qualitative features of the model. We have already addressed a number of related problems within a more consistent theory of particles and their interactions, and this framework provides the support for this model. There remains much theoretical work to be done before the model can be extended to its full domain and be tested against the large number of experimental results accumulated over the years. If the basic model cannot be made to agree with an experiment, there are many variations that may satisfy the requirements.

The points of view we present in this paper are an outgrowth of research motivated primarily by the need to give a better foundation to quantum electrodynamics and other field theories. Unhappiness with the renormalization procedure of quantum field theory made us look for an alternative formulation. An approach that has a number of attractive features is Dirac's many-times formalism of relativistic quantum mechanics (Dirac, 1932). This theory was abandoned in favor of quantum field theory apparently because the creation and annihilation operators of the latter were thought to be required to change the number of particles in a physical state. Nevertheless, we showed (Marx, 1969, 1970a) how pair creation and pair annihilation can be represented in relativistic quantum mechanics by means of an appropriate choice of normalization for the wave function, the specification of initial conditions for particles and final conditions for antiparticles, and a more literal adherence to the Stueckelberg (1941, 1942) and Feynman (1949) interpretation of antiparticles as particles propagating backward in time. Our theory shares many features with the interpretation of the Klein–Gordon equation in relativistic quantum mechanics (Feshbach and Villars, 1958; Bjorken and Drell, 1964). The broader implication of our approach is that elementary-particle physics takes place in a space–time without a preferred time direction, and that human observers introduce the arrow of time because they are made out of matter.

A relativistic wave function can be decomposed into positive- and negative-frequency parts, which represent particles and antiparticles, respectively. The theory of scalar charged particles in external electromagnetic fields follows the familiar pattern of nonrelativistic quantum mechanics, with probability amplitudes and perturbation expansions without divergent terms. We base our interpretation on a conserved charge instead of a conserved probability. The theory of the interaction between nonrelativistic charged particles and a dynamical electromagnetic field suffers from conceptual and practical difficulties. These problems can be traced back to the fact that nonrelativistic quantum mechanics and electromagnetism are invariant under different groups of coordinate transformations and, therefore, that they cannot be properly matched. We have generalized the relativistic quantum mechanics of charged bosons in an external electromagnetic field to the interaction of a single scalar particle with a dynamical electromagnetic field (Marx, 1979), but we have not been able to find a relativistic form of the Coulomb interaction between particles or a full interaction between a many-particle wave function and the electromagnetic field. Our difficulty may go back to the classical theory of charged particles (Rohrlich, 1965; Marx, 1975, 1976), which is not free of inconsistencies. For instance, there are indications in the classical theory (Marx, 1975, 1976) and in the quantum theory (Marx, 1979) that antiparticles interact with advanced electromagnetic fields. We emphasize that we are working with a formalism that describes a fixed number of "particles," which can nevertheless appear both as particles and antiparticles at the same time. As a result, perturbation expansions are free of troublesome closed-loop diagrams.

The theory of charged spin-$\frac{1}{2}$ particles is much less satisfactory. A bispinor field ψ that obeys the Dirac equation represents a spin-$\frac{1}{2}$ particle interacting with an external electromagnetic field. The field ψ forms a conserved "probability current density" j_μ with a positive-definite j_0. The positivity of j_0, which originally was considered an important advantage of the Dirac equation over the Klein–Gordon equation, becomes a drawback in relativistic quantum mechanics. A positive "charge" precludes the interpretation of the wave function in terms of probability amplitudes, essentially because an electron cannot become a positron as time advances. We can change the sign of one or two of the components of the Hamiltonian (Marx, 1970b), much as happens when operators are normal ordered in quantum field theory, but this *ad hoc* procedure is awkward at best. A more satisfactory formulation of the theory is a quantization (Marx, 1972a) of the field ψ of relativistic quantum mechanics that leads to a many-particle formalism very similar to the nonrelativistic theory. There we follow Bopp's suggestion (Bopp, 1965) that antiparticles propagate backward in time when they are created and make all operators in ψ annihilation operators. We further assume that the Hamiltonian operator acts on a state vector to displace particles forward in time and antiparticles backward. The resulting theory has a simple vacuum state that does not suffer from vacuum polarization, has states with a fixed number of "particles," and is equivalent to a relativistic quantum mechanics of many particles. The probability amplitudes are essentially those of Foldy and Wouthuysen (1950), the position operator is $\mathbf{x}$, there is no *zitterbewegung*, and there is no need for an infinite sea of negative-energy electrons. The main disadvantage is that the state vector has one time variable for each "particle."

There has been considerable interest in the two-component spinor formulation of quantum electrodynamics, as opposed to the bispinor or four-component spinor formulation. The Dirac equation can be expressed in both forms, but the bispinor equation can be derived from a Lagrangian density while no such a derivation has been found for the spinor equation. A third-order spinor equation can be obtained from a Lagrangian density and the solutions combine massive and massless particles, but the massless ones are charged (Marx, 1974a). The first-order Weyl equation for massless spinors can also be obtained from a Lagrangian density. We prefer spinor equations because spinors are more fundamental than bispinors, and weak interactions, where parity is not conserved, are more naturally represented in terms of spinors, since spinors that obey the Weyl equation have only one helicity.

We thus come to the conclusion that it is better for fermions to be neutral while bosons carry the charge. Furthermore, if we accept that particles do not appear or disappear, but only go back and forth in time, examination of the diagram for muon decay in Figure 1a suggests that neutrinos, electrons, and muons are in some sense the same particle. There has to be a charge that is originally associated with the muon and that transfers to the electron. Since the pion cannot decay by becoming a virtual nucleon–antinucleon pair because the theory does not allow closed loops, we conclude that the intermediate charged particle is a pion. Hence, we propose that *the electron be considered a bound state of a neutrino and a pion.* And, to make the negative muon different from the electron, we propose

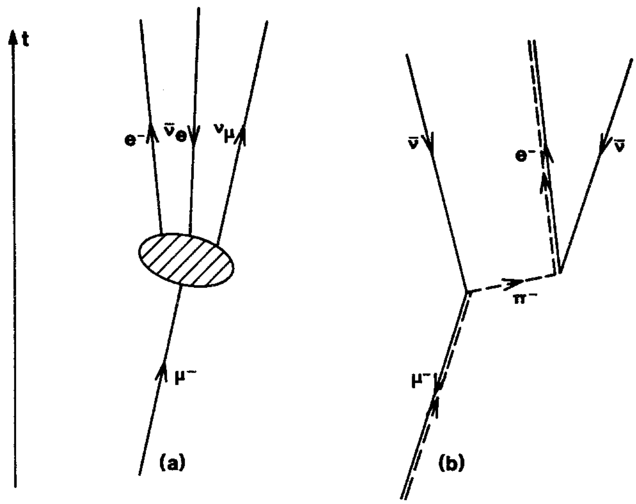

Fig. 1. (a) Traditional view of muon decay. (b) Basic diagram for the new model, showing the composite electron and muon.

that this muon be considered a bound state of an antineutrino and a pion. These assignments of the neutrino and antineutrino may be reversed depending on calculations of the energy of bound states. The new representation of muon decay is shown in Figure 1b. A process that originally involved four different fermions and one intermediate boson is reduced to a process that involves one massless fermion and one massive charged boson. The corresponding reaction is

$$\mu^- \to e^- + 2\bar{\nu} \tag{1}$$

It should be immediately obvious why the reaction

$$\mu^- \to e^- + \gamma \tag{2}$$

is not allowed, since such a decay would require the antineutrino in the muon to become the neutrino in the electron. The pion decay then proceeds according to the diagrams in Figure 2,

$$\pi^- \to \mu^- + \nu \tag{3}$$

$$\pi^- \to e^- + \bar{\nu} \tag{4}$$

Whether the neutrino and the pion retain their identities in these bound states or whether they combine to form new particles is to some extent a matter of definition. A similar question arises when we consider whether mesons and nucleons are composed of quarks and antiquarks.

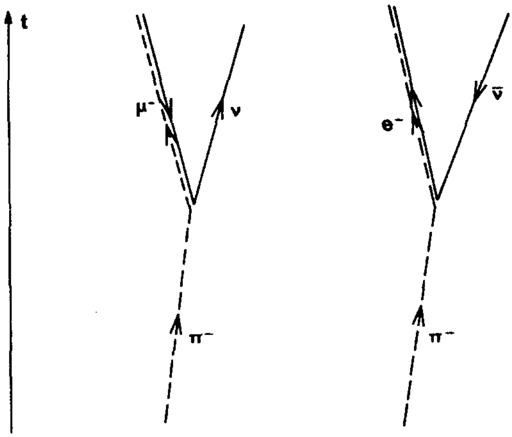

Fig. 2. Pion-decay diagrams according to the new model.

In Section 2 we introduce a Lagrangian density that describes the weak interactions between pions and neutrinos, in Section 3 we add the electromagnetic interactions to this formalism, in Section 4 we present some thoughts on the bound-state problem, and in Section 5 we discuss several alternative formulations.

We use the time-favoring metric in space-time, natural units, and the modified summation convention for repeated lower Greek indices. Notation not defined in this paper is either standard or explained in previous papers.

2. WEAK INTERACTIONS

We choose the spinor field χ_A, $A=1,2$, to represent the neutrino, and the complex (pseudo) scalar field ϕ to represent the pion.

We start from the Lagrangian density

$$\mathscr{L}=\tfrac{1}{2}i(\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_{B,\mu}-\chi^*_{\dot{A},\mu}\sigma_\mu^{\dot{A}B}\chi_B)+\phi^*_{,\mu}\phi_{,\mu}-m^2\phi^*\phi-gj_\mu J_\mu \tag{5}$$

where

$$(\sigma_0^{\dot{A}B})=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad (\sigma_1^{\dot{A}B})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad (\sigma_2^{\dot{A}B})=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$

$$(\sigma_3^{\dot{A}B})=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{6}$$

$$j_\mu=\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_B \tag{7}$$

$$J_\mu=J_\mu^{(0)}-2gj_\mu\phi^*\phi \tag{8}$$

$$J_\mu^{(0)}=i(\phi^*\phi_{,\mu}-\phi^*_{,\mu}\phi) \tag{9}$$

The Euler–Lagrange equations of motion are

$$-i\sigma_\mu^{\dot{A}B}\chi_{B,\mu}+g\sigma_\mu^{\dot{A}B}\chi_B J_\mu^{(0)}=0 \tag{10}$$

$$(\partial^2+m^2)\phi+2igj_\mu\phi_{,\mu}=0 \tag{11}$$

We note that, when $g=0$, equation (10) reduces to the Weyl equation, which describes right-handed massless spin-$\frac{1}{2}$ particles. We have used the identity

$$\sigma_\mu^{\dot{A}B}\sigma_\mu^{\dot{C}D}=2\varepsilon^{\dot{A}\dot{C}}\varepsilon^{BD} \tag{12}$$

where

$$(\varepsilon^{\dot{A}\dot{C}})=(\varepsilon^{BD})=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{13}$$

and conservation of charge,

$$j_{\mu,\mu}=0 \tag{14}$$

in the derivation of equations (10) and (11). In particular, from (12) we derive

$$\sigma_\mu^{\dot{A}B}\chi_B j_\mu=\sigma_\mu^{\dot{A}B}\chi_B\chi_{\dot{C}}^*\sigma_\mu^{\dot{C}D}\chi_D=\chi_B\chi^B\chi^{*\dot{A}}=0 \tag{15}$$

where the matrix in equation (13) is used to raise spinor indices (we recall that $\chi_B\chi^B\equiv 0$), and, consequently, that

$$j^2=0 \tag{16}$$

The pion current density J_μ is also conserved, that is,

$$J_{\mu,\mu}=0 \tag{17}$$

but $J_\mu^{(0)}$ is not conserved. We prefer to use J_μ in the current–current interaction term in the Lagrangian density, although equation (16) implies that the added portion vanishes identically. We thus have two independent conservation laws: (electric) charge is conserved and the number of neutrinos is conserved.

The conserved "charges" are

$$q=\int d^3x\,j_0=\int d^3x\,\chi^\dagger\chi \tag{18}$$

$$Q=\int d^3x\,J_0=\int d^3x[i(\phi^*\dot{\phi}-\dot{\phi}^*\phi)-2g\chi^\dagger\chi\phi^*\phi] \tag{19}$$

The stress–energy tensor obtained from the Lagrangian density (5) is

$$T_{\mu\nu}=\tfrac{1}{2}i(\chi_{\dot{A}}^*\sigma_\mu^{\dot{A}B}\chi_{B,\nu}-\chi_{\dot{A},\nu}^*\sigma_\mu^{\dot{A}B}\chi_B+\phi_{,\nu}^*\phi_{,\mu}+\phi_{,\mu}^*\phi_{,\nu}-gj_\mu J_\nu^{(0)}-\mathscr{L}g_{\mu\nu} \tag{20}$$

The integral of $T_{0\nu}$ over space gives the energy–momentum vector

$$P_0 = \int d^3x(i\chi^*_{\dot{A}}\sigma_i^{\dot{A}B}\chi_{B,i} + \dot{\phi}^*\dot{\phi} + \phi^*_{,i}\phi_{,i} + m^2\phi^*\phi - gj_iJ_i^{(0)}) \tag{21}$$

$$P_i = \int d^3x(i\chi^*_{\dot{A}}\sigma_0^{\dot{A}B}\chi_{B,i} + \phi^*_{,i}\dot{\phi} + \dot{\phi}^*\phi_{,i} - gj_0J_i^{(0)}) \tag{22}$$

where we have used the equations of motion and integration by parts to simplify the expressions for P_μ. (There should be no problem in distinguishing between the index i and the imaginary unit in the first term.)

The plane-wave steady state solutions for the free neutrino are

$$\chi(x) = (2\pi)^{-3/2} b_\lambda(\mathbf{p})\chi^\lambda(\hat{p})\, e^{-ip\cdot x} \tag{23}$$

where the $\chi^\lambda(\hat{p})$ are the helical states

$$\chi^{+1}(\hat{p}) = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\varphi} \end{pmatrix}, \qquad \chi^{-1}(\hat{p}) = \begin{pmatrix} -\sin(\theta/2)\, e^{-i\varphi} \\ \cos(\theta/2) \end{pmatrix}, \tag{24}$$

θ and φ being the polar and azimuthal angles of $\hat{p}$. We set $g = 0$ in equation (10) and substitute the above expression for $\chi(x)$ to obtain

$$p_0 - \lambda|\mathbf{p}| = 0 \tag{25}$$

and, since we assume that $p_0 > 0$, we conclude that

$$\lambda = +1, \qquad p_0 = |\mathbf{p}| \tag{26}$$

For the antineutrino,

$$\chi(x) = (2\pi)^{-3/2} d_\lambda(\mathbf{p})\chi^\lambda(-\hat{p})\, e^{ip\cdot x} \tag{27}$$

whence

$$\lambda = -1, \qquad p_0 = |\mathbf{p}| \tag{28}$$

These solutions are not normalizable and they should be used in wave packets. Nevertheless, we can see that the contribution to the energy is positive for the particle and negative for the antiparticle, although the sign of the energy of the antiparticle may be changed by quantization of the field. The "charge" q is positive for both states.

When the neutrino field interacts with the pion field, the restriction to a single helicity no longer follows from the equation of motion.

The plane-wave states for the free pion are

$$\phi^{(\pm)}(x) = (2\pi)^{-3/2}(2k_0)^{-1/2}\, e^{\mp ik\cdot x} \tag{29}$$

where

$$k_0 = +(\mathbf{k}^2 + m^2)^{1/2} \tag{30}$$

The "charge" Q of these states is positive for the upper signs and negative for the lower ones. The energy is positive for both. Interesting linear combinations of these states are

$$\phi^{(0)}(x) = (2\pi)^{-3/2}(k_0)^{-1/2} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, k\cdot x \tag{31}$$

which have $Q = 0$ and could thus represent the neutral-pion state. This state is similar to a standing wave, and the pion interferes with itself coming and going in time. The neutral pion can then decay into two photons through the electromagnetic interactions just as positronium does. The mass of the neutral pion is reduced by the binding energy to a value close to that of the charged pions.

The angular-momentum tensor density is

$$M_{\mu\nu\lambda} = T_{\mu\nu}x_\lambda - T_{\mu\lambda}x_\nu + S_{\mu\nu\lambda} \tag{32}$$

where

$$S_{\mu\nu\lambda} = \tfrac{1}{2}i\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\mathcal{S}_{\nu\lambda B}{}^C\chi_C - \tfrac{1}{2}i\chi^*_{\dot{C}}\mathcal{S}^*_{\nu\lambda\dot{B}}{}^{\dot{C}}\sigma_\mu^{\dot{B}A}\chi_A \tag{33}$$

$$\mathcal{S}_{\nu\lambda A}{}^B = \tfrac{1}{4}(\sigma_{\nu\dot{C}A}\sigma_\lambda^{\dot{C}B} - \sigma_{\lambda\dot{C}A}\sigma_\nu^{\dot{C}B}) \tag{34}$$

The angular-momentum vector is then

$$M_i = \tfrac{1}{2}\varepsilon_{ijk}\int d^3x\, M_{0jk} = \varepsilon_{ijk}\int d^3x\, T_{0j}x_k + \tfrac{1}{2}\int d^3x\, \chi^*_{\dot{A}}\sigma_i^{\dot{A}B}\chi_B \tag{35}$$

where the second term can be interpreted as the spin angular momentum that comes from the neutrino field.

3. ELECTROMAGNETIC INTERACTIONS

We introduce the electromagnetic interactions by means of gauge-invariant substitution

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \tag{36}$$

but we restrict the electromagnetic gauge transformations to the pion field. (We use the same letters for the modified quantities to keep the notation simple.) The sign of the coefficient of the potentials A_μ reflects the choice of $-e$ for the charge of the particle. The new Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}i(\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_{B,\mu} - \chi^*_{\dot{A},\mu}\sigma_\mu^{\dot{A}B}\chi_B) - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D^*_\mu\phi^*)D_\mu\phi - m^2\phi^*\phi - gj_\mu J_\mu \tag{37}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. The expressions for the densities (7) and (8) remain unchanged, but the current density in equation

(9) is replaced by

$$J_\mu^{(0)} = i[\phi^* D_\mu \phi - (D_\mu^* \phi^*)\phi] \tag{38}$$

The equation of motion (10) for the neutrino remains unchanged, equation (11) is replaced by

$$(D^2 + m^2)\phi + 2igj_\mu D_\mu \phi = 0 \tag{39}$$

and we add the equation for the electromagnetic field,

$$F_{\mu\nu,\nu} = -eJ_\mu \tag{40}$$

When the electromagnetic interactions are included, the neutrino current density j_μ is still conserved and lightlike, and the pion current density J_μ is also conserved. When the pion field vanishes, there is no source for the electromagnetic field. When the neutrino field vanishes, we obtain the usual theory for the charged scalar field. When both fields are present, the source of the electromagnetic field has the additional term shown in equation (8).

4. BOUND-STATE EQUATIONS

A bound state in nonrelativistic quantum mechanics is represented by a stationary solution of the Schrödinger equation that is localized in space. These concepts do not transfer easily to the relativistic theory. In our formulation of relativistic quantum mechanics, we have chosen (Marx, 1972b) quasistationary states of the (modified) Dirac equation to represent the hydrogen atom; these are stationary solutions of the positive-frequency part of the equation. The solution of the full equation then contains a small negative-frequency part, which represents a probability for the electron to be annihilated. This wave function then represents a spin-$\frac{1}{2}$ particle in an external Coulomb field.

Here we are working with two dynamical fields (or three, if we include the electromagnetic interactions), and we want to find bound states for different combinations of positive- and negative-frequency parts. To explore the formulation similar to the relativistic hydrogen atom, we assume that a pion at rest at the origin is represented by

$$\phi(x) = \eta(r) \exp(-ik_0 t) \tag{41}$$

where η is a real function of $r = |\mathbf{x}|$, we find that

$$J_0^{(0)} = 2k_0\eta^2, \qquad J_i^{(0)} = 0 \tag{42}$$

We also assume that also the neutrino is represented mainly by a stationary positive-frequency wave function,

$$\chi(x)=\xi(\mathbf{x})\exp(-ip_0t) \tag{43}$$

The equations of motion (10) and (11) reduce to

$$(-p_0-i\boldsymbol{\sigma}\cdot\nabla+2gk_0\eta^2)\xi(\mathbf{x})=0 \tag{44}$$

$$\left[-k_0^2-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+m^2+2gk_0\xi^\dagger\xi+2ig\xi^\dagger\boldsymbol{\sigma}\cdot\hat{r}\xi\frac{d}{dr}\right]\eta(r)=0 \tag{45}$$

The last term in equation (45) is the only imaginary quantity, whence it has to vanish. The expression (21) for the energy reduces to

$$P_0=\int d^3x(-i\xi^\dagger\boldsymbol{\sigma}\cdot\nabla\xi+k_0^2\eta^2+\eta'^2+m^2\eta^2) \tag{46}$$

We use equation (44) and the real part of equation (45) to express the energy in the form

$$P_0=p_0+k_0-4gk_0\int d^3x\eta^2\xi^\dagger\xi \tag{47}$$

where the wave functions have been normalized to

$$\int d^3x\,\eta^2=(2k_0)^{-1},\qquad \int d^3x\xi^\dagger\xi=1 \tag{48}$$

and we assume that the field η behaves in such a way at 0 and ∞ that the integrated terms vanish when the term η'^2 is integrated by parts. We can apply Schwartz's inequality to the integral in equation (47) to derive

$$P_0\geq p_0+k_0-2g \tag{49}$$

Although equations (44) and (45) are linear in ξ and η, respectively, the system of equations is not linear in both variables. We have to find values of p_0 and k_0 that allow well-behaved nontrivial solutions for ξ and η.

To resolve the angular dependence of ξ, we assume that it is a linear combination of the angular function for total angular momentum $\frac{1}{2}$, namely (Bjorken and Drell, 1964),

$$\mathcal{Y}_1^{1/2}(\theta,\varphi)=(4\pi)^{-1/2}\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\varphi}\end{pmatrix},\qquad \mathcal{Y}_1^{-1/2}(\theta,\varphi)=(4\pi)^{-1/2}\begin{pmatrix}\sin\theta e^{-i\varphi}\\ \cos\theta\end{pmatrix}$$

$$\mathcal{Y}_{-1}^{1/2}(\theta,\varphi)=(4\pi)^{-1/2}\begin{pmatrix}1\\0\end{pmatrix},\qquad \mathcal{Y}_{-1}^{-1/2}(\theta,\varphi)=(4\pi)^{-1/2}\begin{pmatrix}0\\1\end{pmatrix} \tag{50}$$

so that

$$\xi(x)=i^{1/2-m}\eta_{\kappa m}(r)\mathcal{Y}_\kappa^m(\theta,\varphi),\qquad \kappa=1,-1,\qquad m=\tfrac{1}{2},-\tfrac{1}{2}, \tag{51}$$

where the $\eta_{\kappa m}$ are real and θ and φ are the polar and azimuthal angles of $\hat{r}$. These functions satisfy

$$\boldsymbol{\sigma}\cdot\hat{r}\mathcal{Y}_{\kappa}^{m}(\theta,\varphi)=\mathcal{Y}_{-\kappa}^{m}(\theta,\varphi) \tag{52}$$

$$\boldsymbol{\sigma}\cdot\nabla f(r)\mathcal{Y}_{\kappa}^{m}(\theta,\varphi)=\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)f(r)\mathcal{Y}_{-\kappa}^{m}(\theta,\varphi) \tag{53}$$

We can then compute

$$j_0=\eta_{++}^2+\eta_{+-}^2+\eta_{-+}^2+\eta_{--}^2+2\cos\theta(\eta_{++}\eta_{-+}+\eta_{+-}\eta_{--}) \tag{54}$$

$$\begin{aligned}\mathbf{j}\cdot\hat{r}=&\,2(\eta_{++}\eta_{-+}+\eta_{+-}\eta_{--})+\cos\theta(\eta_{++}^2-\eta_{+-}^2+\eta_{-+}^2-\eta_{--}^2)\\&+2\sin\theta\sin\varphi(\eta_{++}\eta_{+-}+\eta_{-+}\eta_{--})\end{aligned} \tag{55}$$

where we have written only the sign of the indices κ and m. We separate the angular dependence and the real and imaginary parts in equation (44) to obtain the radial equations

$$(-p_0+2gk_0\eta^2)\eta_{+,m}+\eta'_{-,m}=0 \tag{56}$$

$$(-p_0+2gk_0\eta^2)\eta_{-,m}-\frac{2}{r}\eta_{+,m}-\eta'_{+,m}=0 \tag{57}$$

We multiply equation (56) by $\eta_{-,m}$, equation (57) by $\eta_{+,m}$, and subtract to find

$$\frac{d}{dr}(\eta_{+,m}^2+\eta_{-,m}^2)+\frac{4}{r}\eta_{+,m}^2=0 \tag{58}$$

We still have to solve equations (45), (56), and (57), the radial current density (55) has to vanish, and the coefficients of the angle-dependent terms of the charge density (54), at least in some approximation. It is not obvious how to do all this. A first guess for η might be

$$\eta(r)=\left(\frac{\lambda}{k_0}\right)^{1/2}\frac{e^{-\lambda r}}{r} \tag{59}$$

which satisfies

$$\nabla^2\eta(r)=\lambda^2\eta(r)+4\pi(\lambda/k_0)^{1/2}\delta(\mathbf{x}) \tag{60}$$

then, equation (45) implies that

$$\lambda^2=m^2-k_0^2 \tag{61}$$

if the interaction terms are neglected. The singularity at the origin is of the type found in the potential for the Coulomb field. The other equations cannot be satisfied exactly, since we would have

$$\eta_{++}\eta_{-+} = \eta_{+-}\eta_{--} = 0 \tag{62}$$

$$\eta_{++}\eta_{+-} + \eta_{-+}\eta_{--} = 0 \tag{63}$$

$$\eta_{++}^2 - \eta_{+-}^2 + \eta_{-+}^2 - \eta_{--}^2 = 0 \tag{64}$$

If $\eta_{++} \neq 0$, it follows that

$$\eta_{-+} = \eta_{+-} = 0 \tag{65}$$

$$\eta_{--} = \pm \eta_{++} \tag{66}$$

Equation (58) then demands that $r^2\eta_{++}$ and η_{--} have to be constant, which contradicts equation (66). The function η in equation (59) could still be substituted into equations (56) and (57) in a successive approximation scheme. We also have to allow for an angular dependence of the pion wave function.

The energy levels of the relativistic hydrogen atom are determined by the requirement that the wave function be normalizable. Here we have to determine both k_0 and p_0 in this manner; sometimes such eigenvalues and eigenfunctions can be found best through an iterative procedure.

The other part of the problem involves the solution of the same set of equations with a negative-frequency wave function ξ. The sign of p_0 in equations (47), (49), (56), and (57) has to be changed.

A completely different approach is required within the context of quantum field theory. Two available formalisms are those of Bethe and Salpeter (1951) and Danos and Gillet (1975).

5. VARIATIONS

We now discuss some related models that also conform to the identification chosen for the leptons. They are more complicated and should be pursued only if necessary.

The neutrino field can be represented by a bispinor field ψ, allowing for both right-handed and left-handed neutrinos. This field can be expressed in terms of two spinor fields, χ and ζ, by setting (Marx, 1974b)

$$\psi(x) = \begin{pmatrix} \chi_A(x) \\ \zeta^{\dot{A}}(x) \end{pmatrix} \tag{67}$$

We change the first term in the Lagrangian density (5) or (37) and write

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}i(\bar{\psi}\gamma_\mu\psi_{,\mu} - \bar{\psi}_{,\mu}\gamma_\mu\psi) + \cdots \\ &= \tfrac{1}{2}i(\chi_{\dot{A}}^*\sigma_\mu^{\dot{A}B}\chi_{B,\mu} - \chi_{\dot{A},\mu}^*\sigma_\mu^{\dot{A}B}\chi_B + \zeta_{\dot{A},\mu}\sigma_\mu^{\dot{A}B}\zeta_B^* - \zeta_{\dot{A}}\sigma_\mu^{\dot{A}B}\zeta_{B,\mu}^*) + \cdots\end{aligned} \tag{68}$$

where the γ matrices are

$$\gamma_\mu = \begin{pmatrix} 0 & -\sigma^*_{\mu A\dot{B}} \\ -\sigma_\mu^{\dot{A}B} & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{69}$$

and

$$\bar{\psi} = \psi^\dagger \gamma_0 \tag{70}$$

The conserved neutrino current density can be a vector,

$$j_\mu = \bar{\psi}\gamma_\mu\psi = \chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_B + \zeta_{\dot{A}}\sigma_\mu^{\dot{A}B}\zeta^*_B \tag{71}$$

or a pseudovector or axial vector

$$j_\mu = i\bar{\psi}\gamma_5\gamma_\mu\psi = \chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_B - \zeta_{\dot{A}}\sigma_\mu^{\dot{A}B}\zeta^*_B \tag{72}$$

In these cases, the density j_μ is no longer lightlike, but

$$j^2 = (\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_B \pm \zeta_{\dot{A}}\sigma_\mu^{\dot{A}B}\zeta^*_B)(\chi^*_{\dot{C}}\sigma_\mu^{\dot{C}D}\chi_D \pm \zeta_{\dot{C}}\sigma_\mu^{\dot{C}D}\zeta^*_D) = \pm 4\chi^*_{\dot{A}}\zeta^{\dot{A}}\chi_B\zeta^{*B} \tag{73}$$

and the equations of motion are changed to

$$-i\sigma_\mu^{\dot{A}B}\chi_{B,\mu} + g\sigma_\mu^{\dot{A}B}\chi_B J'_\mu = 0 \tag{74}$$

$$-i\zeta_{\dot{A},\mu}\sigma_\mu^{\dot{A}B} \pm g\zeta_{\dot{A}}\sigma_\mu^{\dot{A}B}J'_\mu = 0 \tag{75}$$

where

$$J'_\mu = J_\mu - 2gj_\mu\phi^*\phi = J_\mu^{(0)} - 4gj_\mu\phi^*\phi \tag{76}$$

and

$$(D^2 + m^2)\phi + 2igj_\mu D_\mu\phi - 2g^2j^2\phi = 0 \tag{77}$$

while the other two remain unchanged. The projection operators $\frac{1}{2}(1 \pm i\gamma_5)$ can be used to exclude left-handed or right-handed neutrinos from ψ. The field ζ can replace χ throughout Section 2 if left-handed neutrinos are required.

We can also replace the scalar field ϕ with a vector field W_μ, which would be more in agreement with the current view of weak interactions. The Lagrangian density could be

$$\mathscr{L} = \tfrac{1}{2}i(\chi^*_{\dot{A}}\sigma_\mu^{\dot{A}B}\chi_{B,\mu} - \chi^*_{\dot{A},\mu}\sigma_\mu^{\dot{A}B}\chi_B) - W^*_{\mu,\nu}W_{\mu,\nu} + m^2W^*_\mu W_\mu - gj_\mu J_\mu \tag{78}$$

where

$$J_\mu = i(W^*_\alpha W_{\alpha,\mu} - W^*_{\alpha,\mu}W_\alpha) + 2gJ_\mu W^*_\alpha W_\alpha \tag{79}$$

The equation of motion for the vector field is

$$W_{\alpha,\mu\mu} + m^2 W_\alpha - igj_\mu W_{\alpha,\mu} = 0 \tag{80}$$

One problem with the vector field is the elimination of the scalar part, since the subsidiary condition

$$W_{\alpha,\alpha=0} \tag{81}$$

is not compatible with the equation of motion (80). This difficulty is already present in the theory of the electromagnetic interactions of the vector field (Goldberg and Marx, 1967).

6. CONCLUDING REMARKS

In our model of the electromagnetic and weak interactions, we have reduced the number of leptons from four to one, we have indicated how the masses of the electron and the muon should be computed, and we provide a qualitative explanation for the large relative difference between the masses of the muon and the electron. The electron and the muon are bound states of a neutrino and a pion, and the neutral pion is an interference state of the charged pion. The electric charge density is indefinite, as required by the probabilistic interpretation of relativistic quantum mechanics. The neutrino density is positive definite, at least in the simplest form of the theory with no field quantization. If the neutrino field has to be quantized, we avoid the infinities of quantum field theory by modifying the effect of the Hamiltonian operator on the state vector, which has to have one time variable per particle.

The decay of unstable particles, muons and pions, is represented by the creation of a neutrino pair plus the combination of a pion with one of the created neutrinos.

We have not included quarks and Yang–Mills fields in the theory. Beta decay of the neutron is quite compatible with our model, as it can occur through the emission of a virtual pion. Consequently, nucleons and other baryons could be bound states of a basic baryon and one or more pions. The τ particle could be an excited state in the same neutrino–pion system, or it might involve a different type of neutrino.

The bound state problem requires much additional work in relativistic quantum mechanics and in quantum field theory, especially if a mathematically sound version of the latter is developed.

In our model we have only one mass and two coupling constants, and possibly the coupling constants will be found to be related. The masses of the bound states will provide a good test of this theory once they are unambiguously calculated.

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